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CHAOTIC TRAJECTORIES OF A DOUBLE MATHEMATICAL PENDULUM†

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The non-integrability and existence of chaotic trajectories in the high-energy zone are proved for a double mathematical pendulum with certain constraints on the ratio of the masses. © 1999 Elsevier Science Ltd. All rights reserved.

The problem of the integrability and non-integrability of natural Lagrangian systems with a compact two-dimensional configurational manifold M is associated with the Euler characteristic $\chi(M)$. As we know [1, 2], when $\chi(M) < 0$ each analytic first integral depends functionally on the energy integral. However, no such topological obstacle to integrability exists when $\chi(M) = 0$ (M is a two-dimensional torus). The existence of an analytic first integral, independent of the energy integral, for a double mathematical pendulum thus requires further investigation.

The non-integrability of a double mathematical pendulum for energy values close to the maximum potential energy has already been proved [3]. We shall therefore consider a different case. Unlike [3], which uses variational methods, we shall employ the methods of perturbation theory, similar to those used in the four vortices problem [4]. For sufficiently small ratios of the masses and quite high energy values, we will prove that a double pendulum has chaotic trajectories. This result is obtained from an estimate of the Mel'nikov integral.

Numerical evidence of the existence of chaotic motions in both energy regimes was obtained in [5].

Note that the non-existence of an analytical supplementary first integral for a *physical* double pendulum was proved in [6] with certain assumptions.

1. STATEMENT OF THE PROBLEM

A point mass P_1 of mass m_1 moves in a vertical plane along a circle of radius l_1 with center O. The Lagrange coordinate φ_1 is the angle between the vertical and the segment OP_1. The point P_2 of mass m_2 moves in a vertical plane at a constant distance l_2 from P_1. The Lagrange coordinate φ_2 is the angle between OP_1 and P_1P_2. We shall consider "fast motion" of the pendulum, corresponding to high kinetic energy. We make the replacement $\tau = \omega t^{-1/2}$, where ω is a large parameter. Then the Lagrange function takes the form

$$L(\varphi_1, \varphi_2, \dot\varphi_1, \dot\varphi_2) = T(\varphi_2, \dot\varphi_1, \dot\varphi_2) + \frac{g}{\omega} U(\varphi_1, \varphi_2) \tag{1.1}$$

$$T = ((m_1 + m_2)l_1^2 + 2m_2 l_1 l_2 \cos\varphi_2 + m_2 l_2^2)\dot\varphi_1^2/2 + m_2 l_2 (l_2 + l_1 \cos\varphi_2)\dot\varphi_1\dot\varphi_2 + m_2 l_2^2 \dot\varphi_2^2/2$$

$$U = (m_1 + m_2) l_1 \cos\varphi_1 + m_2 l_2 \cos(\varphi_1 + \varphi_2)$$

where the dots denotes differentiation with respect to τ and g is the acceleration due to gravity.

2. THE UNPERTURBED SYSTEM

Ignoring the unimportant multiplier $(m_1 l_1^2)^{-1}$, the Hamilton function corresponding to the Lagrange function (1.1) has the form

$$H = H_0 + \varepsilon H_1 \tag{2.1}$$

$$H_0 = \frac{\mu l^2 p_1^2 - 2\mu l C p_1 p_2 + B p_2^2}{2\mu l^2 A}$$

$$H_1 = -(1+\mu)\cos\varphi_1 - \mu l \cos(\varphi_1 + \varphi_2)$$

$$C = l + \cos\varphi_2, \quad B = 1 + \mu(1 + l^2 + 2l\cos\varphi_2), \quad A = 1 + \mu\sin^2\varphi_2$$

$$l = \frac{l_2}{l_1}, \quad \mu = \frac{m_2}{m_1}, \quad \varepsilon = \frac{l_1^3 m_1^2 g}{\omega}$$

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We shall assume that ε is a small parameter. For the unperturbed Hamiltonian H_0 the generalized momentum p_1 is the first integral.

We fix the positive number E and consider the constraints on the unperturbed system corresponding to the Hamilton function H_0, at energy level $H_0 = E$. By Whittaker's method, we can reduce this system to two Hamiltonian systems with one degree of freedom, corresponding to the Hamilton functions

$$K_0^{\pm}(p_2, \varphi_2, E, \mu) = \frac{\sqrt{\mu} C p_2 \pm \sqrt{A(2E\mu l^2 - p_2^2)}}{l\sqrt{\mu}}$$

defined in the region $D = \{(p_2, \varphi_2) \in R^2 : |p_2| < \sqrt{(2E\mu)}l\}$.

We will consider the Hamiltonian system corresponding to $K_0 = K_0^+$ (the other case is similar)

$$\frac{dp_2}{d\varphi_1} = \frac{\partial K_0}{\partial \varphi_2}, \quad \frac{d\varphi_2}{d\varphi_1} = -\frac{\partial K_0}{\partial p_2}$$

This system has a stable equilibrium position

$$\varphi_2 = 0, \quad p_2 = l(l+1)\mu\sqrt{2E}(1+\mu(l+1)^2)^{-\frac{1}{2}}$$

and hyperbolic equilibria

$$\varphi_2 = \pm\pi, \quad p_2^{\infty} = l(l-1)\mu\sqrt{2E}(1+\mu(l-1)^2)^{-\frac{1}{2}}$$

The hyperbolic positions of equilibrium are connected by separatrices

$$p_2^{\pm} = \frac{\mu l\sqrt{2E}}{B}\left(C\sqrt{1+\mu(l-1)^2} \pm \sqrt{2lAC}\right) \tag{2.2}$$

For sufficiently small μ the separatrices lie in D. To fix our ideas, we will consider the upper separatrix. The equations of motion along it have the form

$$\frac{d\varphi_2}{d\varphi_1} = \frac{B\sqrt{2lC}}{\sqrt{A(1+\mu(l-1)^2)} - \mu\sqrt{2lC^3}}$$

and the solution $\varphi_2(\varphi_1, \delta, \mu)$ is obtained by inverting the integral

$$\varphi_1(\varphi_2, \delta, \mu) = \delta + s(\varphi_2) + O(\mu), \quad s(\varphi_2) = \frac{1}{2\sqrt{l}} \ln\frac{1+\sin(\varphi_2/2)}{1-\sin(\varphi_2/2)} \tag{2.3}$$

where δ is the initial value.

3. THE MEL'NIKOV INTEGRAL

We now consider the complete Hamiltonian (2.1). We will solve the equation $H = E$ for p_1

$$p_1^{\pm} = \frac{\sqrt{\mu} C p_2 \pm \sqrt{A(2(E - \varepsilon H_1)\mu l^2 - p_2^2)}}{l\sqrt{\mu}}$$

The quantity ε must be sufficiently small. By restricting the Hamiltonian system to energy level $H = E$ in the neighbourhood of separatrix (2.2) we can reduce the system to one-and-a-half-degrees-of-freedom system with the Hamilton function

$$K(p_2, \varphi_2, \varphi_1, E, \mu, \varepsilon) = K_0(p_2, \varphi_2, E, \mu) + \varepsilon K_1(p_2, \varphi_2, \varphi_1, E, \mu) + \ldots$$

$$K_1(p_2, \varphi_2, \varphi_1, E, \mu) = -H_1(\varphi_1, \varphi_2, \mu)\partial K_0/\partial E$$

Representing separatrix (2.2) of the unperturbed system in the form

$$p_2 = p_2(\varphi_2, \mu), \quad \varphi_2 = \varphi_2(\varphi_1, \delta, \mu)$$

we write the Mel'nikov integral

$$M(\mu,\delta)=\int_{-\infty}^{+\infty}\left\{\frac{\partial K_0}{\partial\varphi_2}\frac{\partial K_1}{\partial p_2}-\frac{\partial K_0}{\partial p_2}\frac{\partial K_1}{\partial\varphi_2}\right\}_{\substack{p_2=p_2(\varphi_2,\mu)\\ \varphi_2=\varphi_2(\varphi_1,\delta,\mu)}}d\varphi_1$$

Writing the expression in braces in the form

$$\left(\frac{\partial K_0(p_2,\varphi_2,E,\mu)}{\partial E}-\frac{\partial K_0(p_2^\infty,\pi,E,\mu)}{\partial E}\right)\frac{\partial H_1(\varphi_1,\varphi_2,\mu)}{\partial\varphi_1}+$$

$$+\frac{\partial K_0(p_2^\infty,\pi,E,\mu)}{\partial E}\left(\frac{\partial H_1(\varphi_1,\varphi_2,\mu)}{\partial\varphi_1}-\frac{\partial H_1(\varphi_1,\pi,\mu)}{\partial\varphi_1}\right)$$

and using explicit expressions for K_0 and H_1, we obtain

$$M=\mu\sqrt{\frac{2}{E}}\,I\sin\delta+o(\mu)$$

$$I=\int_0^\pi\left(\frac{3}{2}\sqrt{2l(1+\cos x)}+l-1+\cos x\right)\cos s(x)\,dx$$

Using Eqs (2.3) and

$$\int_0^\pi\cos x\cos s(x)\,dx=\frac{1}{l}\int_0^\pi\cos s(x)\,dx$$

$$\int_0^\pi\sqrt{2l(1+\cos x)}\cos s(x)\,dx=2l\int_0^\pi\sin x\sin s(x)\,dx$$

we obtain

$$I=3lI_1+(l-1+l^{-1})I_2$$

$$I_1=2\,\mathrm{Im}\int_{-\infty}^{+\infty}e^{iku}\frac{\mathrm{sh}\,u}{\mathrm{ch}^3u}du,\quad I_2=\mathrm{Re}\int_{-\infty}^{+\infty}\frac{e^{iku}}{\mathrm{ch}\,u}du;\quad k=\frac{1}{\sqrt{l}}$$

We will evaluate the integrals I_1 and I_2 by means of residues, taking the integration contour as a small circle with centre $z=0$ inside the strip $\{z\in C:\mathrm{Im}\,z\in[-\pi/2,\pi/2]\}$. Finally, we find

$$I=2\pi\left(3\,\mathrm{ch}\frac{\pi}{2\sqrt{l}}+\frac{l^2-l+1}{l\sqrt{l}}\mathrm{sh}\frac{\pi}{2\sqrt{l}}\right)\Big/\mathrm{sh}\frac{\pi}{\sqrt{l}}$$

Since $I>0$ when $l>0$, we have the following theorem.

Theorem. For any $l>0$ a $\mu(l)>0$ exists and there is an analytic function $\delta:(0,\mu(l)\to R$, $\lim_{\mu\to 0}\delta\mu=0$ for which $\delta(\mu)$ is a simple zero of the Mel'nikov integral.

Hence at very high energies, that is, for sufficiently small ε, there is a transversal homoclinic point ν_ε, that is, a point of the transversal intersection of the stable and unstable manifolds of the Poincaré image Ψ_ε of the system with Hamilton function H. Hence, there is some power of Ψ_ε on an invariant subset in the neighbourhood of the set $\overline{\cup_{n\in Z}\Psi_\varepsilon^n(\nu_\varepsilon)}$ which is conjugate to a Bernoulli shear on the set of infinite sequences of two symbols [7, 8]. In particular, the system has no analytic first integrals which are functionally independent of the energy integral.

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REFERENCES

1. KOZLOV, V. V., *Symmetry, Topology and Resonances in Hamiltonian Mechanics*. Izd. Udm. Univ., Izhevsk, 1995.
2. BOLOTIN, S. V., Homoclinic orbits of geodesic flows on surfaces. *Russian J. Math. Phys.*, 1993, **1**, 275–288.
3. BOLOTIN, S. V. and NEGRINI, P., A variational criterion for non-integrability. *Russian J. Math. Phys.*, 1997, **5**, 415–439.
4. CASTILLA, M. S., MOAURO, V., NEGRINI, P. and OLIVA, W. M., The four positive vortices problem: Regions of chaotic behaviour and the non-integrability. *Ann. Inst. Poincaré. Phys. Théor*, 1993, **59**, 99–115.

5. RICHTER, P. H. and SCHOLZ, H. J., Chaos in classical mechanics: The double pendulum. In: *Stochastic Phenomena and Chaotic Behaviour in Complex Systems*, 86–96. Springer, Berlin, 1984.
6. BUROV, A. A., On the non-existence of a supplementary integral in the problem of a heavy double plane pendulum. *Prikl. Mat. Mekh.*, 1986, **50**, 168–171.
7. MOSER, J., *Stable and Random Motions in Dynamical Systems. Ann. Math. Studies*. University Press, Princeton, Vol. 77, 1973.
8. SMALE, S., Diffeomorphisms with many periodic points. In *Differential and Combinatorial Topology*. University Press, Princeton. Vol. 17, 63–80, 1965.

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